A bicycle can be self-stable without gyroscopic or caster effects

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A riderless bicycle can automatically steer itself so as to recover from falls. The common view is that this self-steering is caused by gyroscopic precession of the front wheel, or by the wheel contact trailing like a caster behind the steer axis. We show that neither effect is necessary for self-stability. Using linearized stability calculations as a guide, we built a bicycle with extra counter-rotating wheels (canceling the wheel spin angular momentum) and with its front-wheel ground-contact forward of the steer axis (making the trailing distance negative). When laterally disturbed from rolling straight this bicycle automatically recovers to upright travel. Our results show that various design variables, like the front mass location and the steer axis tilt, contribute to stability in complex interacting ways.

A bicycle and rider in forward motion balance by steering towards a fall, which brings the wheels back under the rider (*1*) (also see Ch. S1-2). Normally riders turn the handlebars with their hands to steer for balance. With hands off the handlebars, body leaning relative to the bicycle frame can also cause appropriate steering. Amazingly, many moving bicycles with no rider can steer themselves so as to balance; likewise with a rigid rider whose hands are off the handlebars. For example, in 1876 Spencer (*2, 3*) noted that one could ride a bicycle while lying on the seat with hands off, and the film *'Jour de fête'* by Jacques Tati, 1949, features a riderless bicycle self-balancing for long distances. Suspecting that bicycle rideability with rider control is correlated with self-stability of the passive bicycle, or at least not too much self-instability, much theoretical research has focused on this bicycle self-stability.

The first analytic predictions of bicycle self-stability were presented independently by French mathematician Emmanuel Carvallo (*4*) (1897) and Cambridge undergraduate Francis Whipple (*3, 5*) (1899). In their models, and in this paper, a bicycle is defined as a three-dimensional mechanism (Fig. 1A) made up of four rigid objects (the rear frame with rider body B, the handlebar assembly H, and two rolling wheels R and F) connected by three hinges. The more complete Whipple version has 25 geometry and mass parameters. Assuming small lean and steer angles, linear and angular momentum balance, as constrained by the hinges and rolling contact, lead to a pair of coupled second-order linear differential equations for leaning and steering (*6*) (see also Ch. S3). Solutions of these equations show that after small perturbations the motions of a bicycle may exponentially decay in time to upright straight-ahead motion (asymptotic stability). This stability typically can occur at forward speeds v near to $\sqrt{gL}$, where g is gravity and L is a characteristic length (about 1m for a modern bicycle). Limitations in the model include assumed linearity and the neglect of motions associated with tire and frame deformation, tire slip, and play and friction in the hinges. Nonetheless, modern experiments have demonstrated the accuracy of the Whipple model for a real bicycle without a rider (*7*).

The simple bicycle model above is energy-conserving. Thus the asymptotic stability of a bicycle, that the lean and steer angles exponentially decay to zero after a perturbation, is jarring to those familiar with Hamiltonian dynamics. But because of the rolling (non-holonomic) contact of the bicycle wheels, the bicycle, although energy conserving, is not Hamiltonian and it is possible for a subset of variables to have exponential stability in time (*6, 8*). There is no contradiction between exponential decay and energy conservation because for a bicycle the energy lost from decaying steering and leaning motions goes to increase the forward speed. Unresolved, however, is the cause of bicycle self-stability. In some sense, perhaps, a self-stable bicycle is something like a system with control, albeit self-imposed.

Rider-controlled stability of bicycles is indeed related to their self-stability. Experiments like those of Jones (*9*) and R. Klein (*10*) show that special experimental bicycles that are difficult for a person to ride, either with hands on or off, tend not to be self-stable. Both no-hands control (using body bending) and bicycle self-stability depend on 'cross terms' in which leaning causes steering or *vice versa*. The central question about what causes self-stability is thus reduced to: what causes the appropriate coupling between leaning and steering? The most often discussed of the coupling effects are those due to front-wheel *gyroscopic torque* and to *caster* effects from the front wheel trailing behind the steer axis. Trail (or 'caster trail') is the distance c that the ground contract point trails behind the intersection of the steering axis with the ground (see Fig. 1A).

There is near universal acceptance that either spin angular momentum (gyroscopic torque) or trail, or both, are necessary for bicycle stability (*3*). These two effects are discussed below, in order, and then considered more critically. Active steering of a bicycle front wheel causes a gyroscopic torque on an upright frame and rider. Because the front wheel is relatively light compared to the more massive bicycle and rider, the effect of this gyroscopic torque on the lean is generally small (*11*) (see also Ch. S1). However, coupling the other way, i.e., the effect of active bicycle leaning on hands-free steering, is non-negligible. For example, when the bicycle has a lean rate to the right, the front axle also has a lean rate

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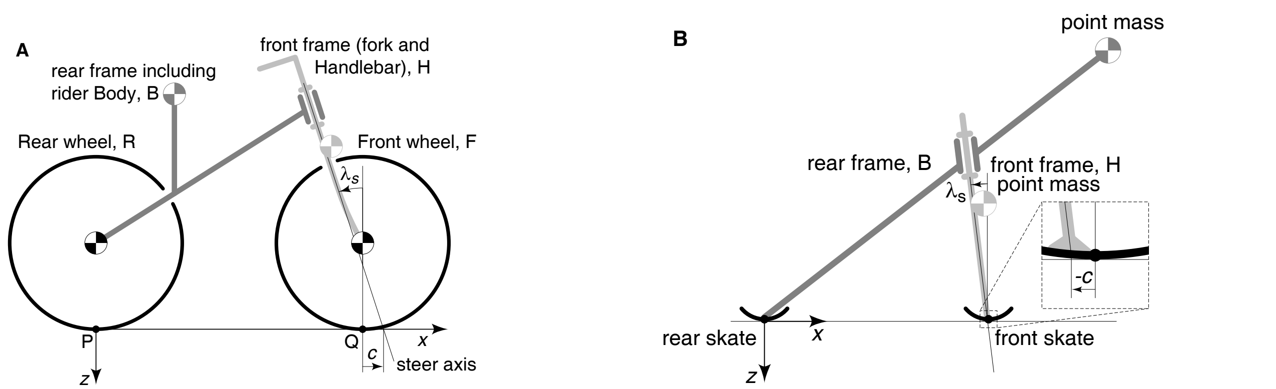

Figure 1: (**A**) The bicycle model consists of two interconnected frames B and H connected to two wheels R and F. The model has a total of 25 geometry and mass-distribution parameters. Central here are the rotary inertia I_{yy} of the front wheel, the steer axis angle ('rake') λ_s and the trail distance c (positive if contact is behind the steer axis). Depending on the parameter values, as well as gravity g and forward speed v, this bicycle can be self-stable or not. (**B**) A theoretical two-mass-skate (TMS) bicycle is a special case described with only 9 free parameters (8 + trail). The wheels function effectively as ice-skates. The two frames each have a single point mass and no mass moments of inertia. A heavy point mass on the rear frame at the rear skate ground contact point can prevent the bicycle from tipping over frontward; because it has no effect on the linearized dynamics it is not shown. Even with negative trail ($c < 0$, see inset) this non-gyroscopic bicycle can be self-stable.

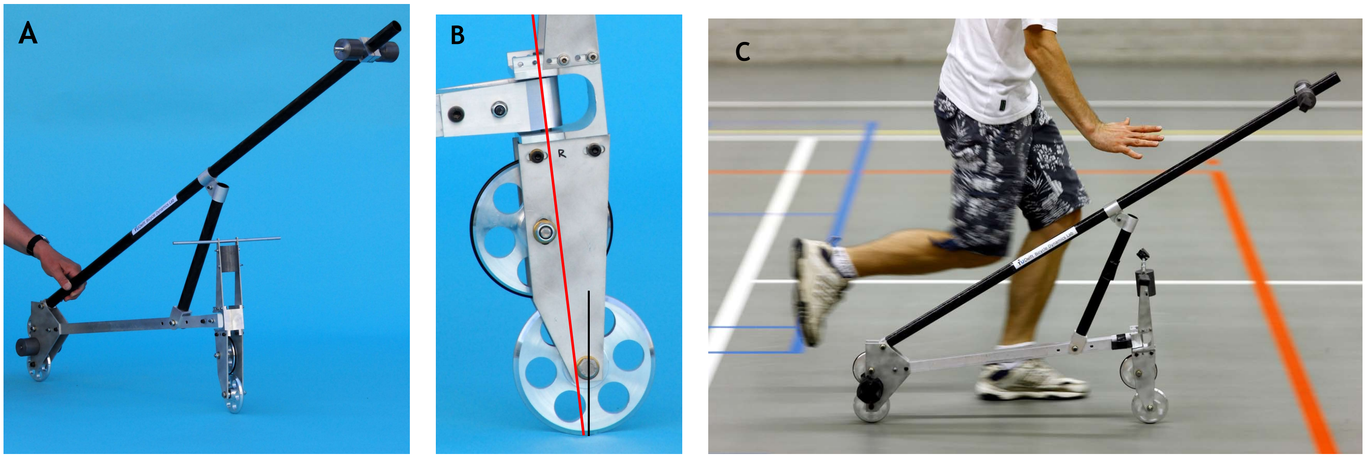

Figure 2: Realization of the model from Fig. 1B. (**A**) The experimental two-mass-skate (TMS) bicycle. (**B**) Front assembly. A counter-rotating wheel cancels the spin angular momentum. The ground contact is slightly ahead of the intersection of the long steer axis line with the ground, showing a small negative trail (Video S3). (**C**) Self-stable experimental TMS bicycle rolling and balancing (photo C by Sam Rentmeester/FMAX).

to the right, and the spinning wheel exerts a clockwise (looking down) reactive torque carried, at least in part, by the handlebar assembly. This reaction torque tends to turn the handlebars rightward. Thus the common explanation of no-hands rider control: to steer to the right, the rider bends her upper body to the left, tilting the bicycle and wheels rightward (*5*). The bicycle handlebars, considered as freely rotating on the steer axis and forced by the gyroscopic front wheel, thus initially turn rightward. Such leaning-induced steering can be used for rider control of balance. Likewise, this gyroscopic coupling also contributes to a forward-moving passive bicycle self-steering toward a fall (*12*).

The most thorough discussion of the necessity of gyroscopic coupling of leaning to steering for bicycle self-stability is in the bicycle chapter of the fourth volume of the gyroscope treatise by Klein and Sommerfeld (*11*) (K&S). They took the example bicycle parameters from Whipple and eliminated just the spin angular momentum of the wheels. Using their own linearized dynamic stability analysis of the Whipple model, K&S concluded that "... in the absence of gyroscopic actions, the speed range of complete stability would vanish" (*11*) and make what appears to be a strong general claim about bicycles:

> "The gyroscopic action, in spite of its smallness, is necessary for self-stability." (p. 866 (*11*))

They emphasized that the gyroscopic torque does not apply corrective lean torques to a bicycle directly, as others seem to have thought (*13*). Rather, leaning causes, through the gyroscopic torque, steering, which in turn causes the righting accelerations: "The proper stabilizing force, which overwhelms the force of gravity, is the centrifugal force, and the gyroscopic action plays the role of a trigger." (*11*)

In Jones's famous search for an unrideable bicycle (URB) (*9*), he added a counter-rotating disk to the handlebar assembly, canceling the gyroscopic self-steering torque of the front wheel. He could still (barely) ride such a non-gyro bicycle no-hands. Jones rightly deduced that the gyroscopic effect discussed in K&S was not the only coupling between leaning and steering. Jones emphasized the importance of the front-wheel ground contact being behind the steering axis (i.e., positive trail, $c > 0$, Fig. 1A). Even though the front forks of modern bicycles are typically bent forward slightly, with the wheel-center forward of the steering axis, all modern bicycles still have positive trail (typically from 2 - 10 cm) because of the steering axis tilt, $\lambda_s > 0$. When Jones modified his bicycle by placing the front-wheel ground contact in front of the steer axis (negative trail, $c < 0$) he could not ride no-hands.

In Jones's view a bicycle wheel is, in part, like a caster wheel on a shopping cart, where the wheel trails behind a vertical pivot axis. If a modern bicycle is rolled forward by guiding the rear frame in a straight line while it is held rigidly upright, the front wheel will quickly self-center like a shopping-cart caster. Jones noted "The bicycle has only geometrical caster [trail] stability to provide its self-centering". Jones's main focus was a second trail effect: the vertical ground contact force on the front wheel ground contact point exerts a steering torque on a leaned bicycle even when the bicycle is steered straight. Jones calculated the steer torque caused by lean as a derivative of a static potential energy, neglecting the weight of the front assembly. If a typical modern bicycle is firmly held by the rear frame, leaned to the right, and pressed down hard, then the vertical ground contact forces on the front wheel cause a rightward steering torque on the handlebars. The Jones torque can be felt on a normal bicycle by riding in a straight line and bending your upper body to the left, leaning the bicycle to the right: to maintain a straight path the hands must fight the Jones torque and apply a leftwards torque to the handlebars. According to Jones, this torque causes steering toward a fall only when the trail is positive. When the trail is zero, Jones's theory predicts no self-correcting steer torque. Jones seems to conclude that no-hands control authority (the ability to cause steering by body bending) and self-stability both depend on positive trail. A mixture of the two mechanisms Jones discusses certainly suggests that trail is a key part of bicycle stability.

Following K&S and Jones, it has become common belief that steering is stable because the front wheel ground contact drags behind the steering axis, and leaning is stable because some mixture of gyroscopic torques and trail cause an uncontrolled bicycle to steer in the direction of a fall (*3*).

Are gyroscopic terms or positive trail, together or separately, really either necessary or sufficient for bicycle self-stability? Following Carvallo, Whipple, K&S and others since (see history in (*6*)) we start with the linearized equations of motion. Using the numerical values from the benchmark example in (*6*)) and setting the gyroscopic terms to zero we find here that self-stability is lost (Ch. S6.1, similar to the result of K&S for the Whipple parameters). However, we also found bicycle designs that are self-stable without gyroscopic terms.

The conflict with K&S is partly resolved by noting sign errors in their key stability term (*3*). Despite their calculation errors, the Whipple bicycle, with Whipple's example parameters, does indeed lose self-stability when the gyro terms are set to zero. But with their incorrect expressions, K&S could make slightly more general claims that are not valid when the sign errors are corrected (*3*). Whatever generality K&S intended (their wording is ambiguous), their result does not apply to bicycles in general.

Similarly Jones's simplified static energy calculation seems incomplete in the context of a dynamical system, like the Whipple and Carvallo models. Jones's static energy calculation only calculates (incompletely) one term, $K_{0\delta\phi}$, of the full dynamics equations (*3, 6*). In a full dynamic analysis $K_{0\delta\phi}$ does not predict the steering of a falling bicycle (*3*). For example, that term can be non-zero for a bicycle that falls with no self-corrective steering at all. And, just as for the gyroscopic term, we can find designs with zero or negative trail that we predict are self-stable (Ch. S6.2).

In contrast to the conventional claims above for the necessity of gyroscopic terms and trail, we have found no rigorous reasoning that demands either. To understand better what is needed for self-stability, we eliminated as many bicycle parameters as possible (*14*). Most centrally, we eliminated the gyroscopic terms and set the trail to zero ($c = 0$). We also reduced the mass distribution to just two point masses: one for the rear frame B and one for the steering assembly H (Fig. 1B). With these theoretical parameters the wheels having no net spin angular momentum, are mechanically equivalent to skates. These simplifications reduce the number of parameters from Whipple's 25 to a more manageable 8.

Stability analysis of this theoretical two-mass-

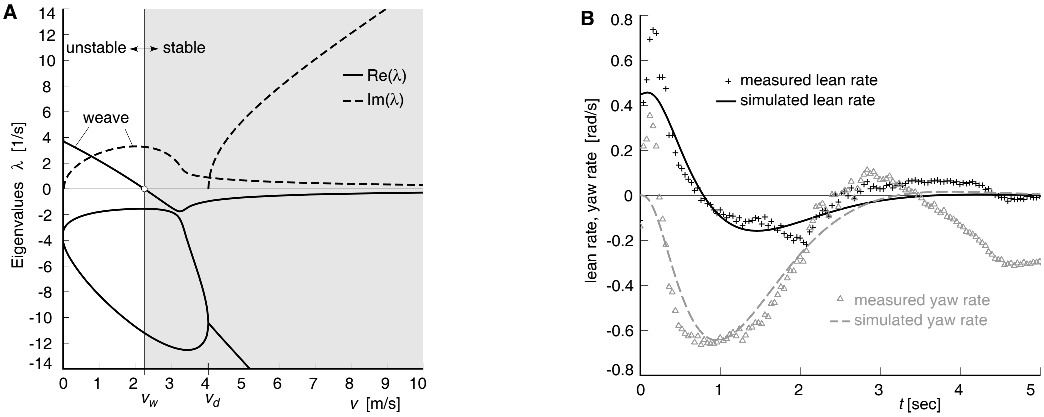

Figure 3: (**A**) Stability plot for the experimental TMS stable bicycle. Solutions of the differential equations are exponential functions of time. Stability corresponds to all such solutions having exponential decay (rather than exponential growth). Such decay only occurs if all four of the eigenvalues λ_i (which are generally complex numbers) have negative real parts. The plot shows calculated eigenvalues as a function of forward speed v. For $v > 2.3$ m/s (the shaded region) the real parts (solid lines) of all eigenvalues are negative (below the horizontal axis) and the bicycle is self-stable. (**B**) Transient motion after a disturbance for the experimental TMS bicycle. Measured and predicted lean and yaw (heading) rates of the rear frame are shown. The predicted motions show the theoretical (oscillatory) exponential decay. Not visible in these plots, but visible in high-speed video (Video S4), is a 20 Hz shimmy that is not predicted by the low-dimensional linearized model (Ch. S14-15).

skate (TMS) bicycle model (Ch. S7), confirmed by numerical solution of the governing differential equations (Fig. 3B), shows that neither gyroscopic terms nor positive trail are needed for self-stability (Routh-Hurwitz (*15*) analysis shows that all eigenvalues of the theoretical TMS bicycle can have negative real parts at some forward speeds, Fig. 3A).

We used the stable theoretical TMS bicycle parameters as a basis for building an experimental TMS bicycle (Fig. 2A, Ch. S8-9). We used small wheels to minimize the spin angular momentum. To further reduce the gyroscopic terms, following Jones, we added counter-spinning disks that rotate backward relative to the lower wheels (Fig. 2B, video S2). The experimental TMS bicycle was built to have a slightly negative trail ($c = -4$ mm < 0, Video S3). While the experimental TMS bicycle looks like a folding scooter, it is still a bicycle (two wheels, two frames, three hinges).

Because all physical objects have distributed mass, the measured parameters of the experimental TMS bicycle were necessarily slightly different from those of the theoretical design, which was based on point masses. Using measured parameters, we calculate the stability plot of Fig. 3A (Ch. S7-8). For rolling speeds greater than 2.3 m/s all eigenvalues have negative real parts (implying self-stability).

After an initial forward push, the coasting experimental TMS bicycle (Fig. 2C) would remain upright before it slowed down to about 2 m/s (Video S1, Ch S10-11). As it slowed down below 2 m/s the bicycle would begin to fall. In a perturbation experiment, the stable coasting bicycle ($v > 2.3$ m/s) was hit sideways on the frame, causing a jump in the lean rate, followed by a recovery to straight-ahead upright rolling.

The lean and yaw rates were measured (telemetered). A data set is compared to theory in Fig. 3B (Video S4). One difference between experiment and theory is lateral wheel slip at the initial perturbation, which caused an initial jump in the measured yaw rate (triangles in the first 0.25 s of Fig. 3B). The theoretical model assumed no slip. High-speed video (Video S4) also shows a 20 Hz shimmy, which is due, at least in part, to unmodeled steering axis play (Ch. S11). Nonetheless, after the slipping period, even with the shimmy, the data reasonably track the low-dimensional linear model's predictions.

Both the theoretical analysis and physical experiment show that neither gyroscopic torques nor trail are necessary for bicycle self-stability. Nor are they sufficient. Many bicycle designs with gyroscopic front wheels and positive trail are unstable at every forward speed (Ch. S6.3). Also, all known bicycle and motorcycle designs lose self-stability at high speeds *because* of gyroscopic terms (e.g. (*6*)). In contrast the TMS bicycle does not have gyroscopic terms and is predicted to maintain stability at high speeds.

With no gyroscopic torque and no trail, why does our experimental TMS bicycle turn in the direction of a fall? A general bicycle is complicated, with various terms that can cause the needed coupling of leaning to steering. Only some of these terms depend on positive trail or on positive spin angular momentum in the front wheel. In the theoretical and experimental TMS designs, the front assembly mass is forward of the steering axis and lower than the rear-frame mass. When the TMS bicycle falls, the lower steering-mass would, on its own, fall faster than the higher frame-mass for the same reason that a short pencil balanced on end (an inverted pendulum) falls faster than a tall broomstick (a slower inverted pendulum). Because the frames are hinged together, the tendency for the front steering-assembly mass to fall faster causes steering in the fall direction. The importance of front assembly mass for Jones-like static torques has been noted before (*8, 16, 17*).

Why does this bicycle steer the proper amounts at the proper times to assure self-stability? We have found no simple physical explanation equivalent to the mathematical statement that all eigenvalues must have negative real parts (Ch. S4).

For example, turning toward a fall is not sufficient to guarantee self-stability. For various candidate simple sufficient conditions X for stability, we have found designs that have X but that are not self-stable. For example, we have found bicycles with gyroscopic wheels and positive trail that are not stable at any speed (Ch. S6.3). We also have found no simple necessary conditions for self-stability. Besides the design with no gyroscope and negative trail we have found other counter-examples to common lore. We have found a bicycle that is self-stable with rear-wheel steering (Ch. S6.7). We also found an alternative theoretical TMS design that has, in addition to no-gyro and negative trail, also a negative head angle ($\lambda_s < 0$, Ch. S6.6).

Are there any simply described design features that are universally needed for bicycle self-stability? Within the domain of our linearized equations, here is one simple necessary condition we have found (Ch. S5):

> *To hold a self-stable bicycle in a right steady turn requires a left torque on the handlebars.*

Equivalently, if the hands are suddenly released from holding a self-stable bicycle in a steady turn to the right, the immediate first motion of the handlebars will be a turn further to the right. This is a rigorous version of the more general as-yet-unproved claim that a stable bicycle must turn toward a fall.

Another simple necessary condition for self-stability is that at least one factor coupling lean to steer must be present (at least one of $M_{\delta\phi}, C_{\delta\phi}$, or $K_{\delta\phi}$) must be non-zero, Ch. S3). These coupling terms arise from combinations of trail, spin momentum, steer axis tilt, and center of mass locations and products of inertia of the front and rear assemblies.

Although we showed that neither front-wheel spin angular momentum nor trail are necessary for self-stability, we do not deny that both are often important contributors. But other parameters are also important, especially the front-assembly mass distribution, and all the parameters interact in complex ways. As a rule we have found that almost any self-stable bicycle can be made unstable by mis-adjusting only the trail, or only the front-wheel gyro, or only the front-assembly center-of-mass position. Conversely many unstable bicycles can be made stable by appropriately adjusting any one of these three design variables, sometimes in an unusual way. These results hint that the evolutionary, and generally incremental, process that has led to common present bicycle designs might not yet have explored potentially useful regions in design space.

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Supporting Online Material Chapters S1-11 include details of the theory, design, construction and testing. Videos S1-4 show the experimental TMS bicyle in motion and some experimental details.